Scientific Report on the implementation of the project
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Time optimal controllability for infinite dimensional systems

Stage 1. The establishment of new regularity results for the cost functions associated to a linear system.

In the frame of the present project the following activities took place: documentation, updating the bibliography, scientific contacts, analysis, research and establishing preliminary results. Also, we have published a paper in an ISI journal. The proposed objectives were completely realized.

In the following we give a presentation of the main results obtained.

Let X be a real Banach space and $A : D(A) \subset X \to X$ the infinitesimal generator of a C_0-semigrup $\{S(t) : X \to X;\, t \geq 0\}$. We consider the linear control system

$$y'(t) = Ay(t) + Bu(t), \ t \geq 0, \tag{1}$$

where $B : X \to U$ is a linear operator, U a Banach space, and the control u is from $L^\infty(0, t; U)$, $t > 0$

We are interesting in obtaining new regularity results for the cost functions associated to 1. The cost functions considered are the minimum time function, i.e. the time taken to steer an initial state of the system to a given target by bounded controls, and the minimum control costs, i.e., the norm of the control by which a given initial state can be driven to a given target within a fixed time interval.

The properties of the minimum time function provide the basis of the dynamic programming method in optimal control, the minimum time function plays an essential role in the study of Hamilton-Jacobi-Bellman equation. In what follows we recall some important papers where there was studied the regularity of the minimum time function, being obtained results on the continuity, Lipschitz or Holder continuity, semiconcavity or monotonicity of the minimum time function, under different frames and hypothesis. We mention [Wo-Z], [C-N], [No], [N-S1], [OC84] for Lipschitz regularity results, [C-S], [C], [S], [C-Ca], [N-S2] for semiconcavity results. In the papers [CL-dcds], [CL-an], there was established Holder continuity for the minimum time function in the case when the controls take values in a smaller space than the space the system is modeled on. In [GL] there was proved the continuity of the minimum control cost and minimum time function and the monotonicity of the minimum control cost, for distributed control systems in Hilbert spaces. H. O. Fattorini [F] established results on the minimum time and minimum norm problems for full control systems ($B = I$). A semilinear system of the form $y'(t) = Ay(t) + f(y(t)) + u(t)$ was studied in [OC 2006] and regularity results on the minimum time function as well as the minimum energy were obtained. In [WZ], there is proved that the minimum time function and the control cost, associated to the heat equation with internally distributed controls, are strictly decreasing and continuous.

In the frame of this research project we consider the linear control system

$$y(t, x, u) = S(t)x + V(t)u$$

in a Banach space X.

Here, $\{S(t); t \geq 0\}$ is a C_0-semigroup on X and $V(t), t > 0$, is a family of bounded linear operators, $V(t) : L^\infty(0,t;U) \to X$, U a Banach space, such that the following condition is satisfied: $V(t_1+t_2)u = S(t_2)V(t_1)u + V(t_2)J_{t_1}u$, for all $t_1, t_2 > 0$ and $u \in L^\infty(0, t_1+t_2; U)$, where J_{t_1} is a translation operator.

The basic hypothesis we shall refer to in the sequel is that:

(H) there exists $\gamma : (0,+\infty) \to (0,+\infty)$ such that

$$S(t)B(0,\gamma(t)) \subseteq V(t)B(0,1), \tag{2}$$

for any $t > 0$.

This condition represents the null controllability for arbitrary short time. As a matter of fact, as results from [CL], if such a function γ exists on some interval $(0,T]$, then it can be extended to the whole $\mathbb{R}_+$. By the open mapping theorem, (2) is equivalent to

$$Range\left(S\left(t\right)\right) \subseteq Range\left(V\left(t\right)\right),$$

which means that all points of X can be transferred to zero in time t by $L^\infty\left(0,t;U\right)$-controls.

We have obtained the local uniform continuity of the minimum time function on the reachable set.

Teorema 1 *Assume (H) with γ continuous, strictly increasing and $\gamma(0) = 0$. If $M_\gamma < +\infty$, then $\mathcal{R}_r$ is open and for any $x_0 \in \mathcal{R}_r$ we have*

$$\left|\mathcal{T}\left(r,z_1\right) - \mathcal{T}\left(r,z_2\right)\right| \leq \gamma^{-1}\left(c_r \left\|z_1 - z_2\right\|\right) \tag{3}$$

for any $z_1, z_2 \in B\left(x_0, \delta_r\right)$, where

$$c_r = \frac{M}{r} e^{\omega\left(\mathcal{T}(r,x_0)+\gamma^{-1}(M_\gamma/k)\right)} \text{ and } \delta_r = \frac{rM_\gamma}{Mk} e^{-\omega(\mathcal{T}(r,x_0)+M_\gamma)}$$

for some $k \geq 2$.

Also, we have proved the Lipschitz continuity of the minimum energy function in the state variable.

Teorema 2 *Assume (H). Then, for every $x, z \in X$ and every $t > 0$, we have*

$$\left|\mathcal{E}\left(t,x\right) - \mathcal{E}\left(t,z\right)\right| \leq \frac{1}{\gamma\left(t\right)} \left\|x - z\right\|.$$

Regarding the monotonicity of the minimum time function we have the following result.

Teorema 3 *Assume (H) and that, for any $u \in U$, $t \mapsto V(t)u$ is continuous in zero. Then the function $\mathcal{T}(\cdot, x)$ is monotonically decreasing for any $x \in X$. In addition, assume the existence of time optimal controls. If $0 < r_1 < r_2$ and $x \in \mathcal{R}_{r_1}$, then*

$$\mathcal{T}\left(r_2, x\right) < \mathcal{T}\left(r_1, x\right).$$

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In **2018** it was published **1 article (**in **ISI journal)**:

S. Bilal, O. Cârjă, T. Donchev, A. I. Lazu, Nonlocal problem for evolution inclusions with one-sided Perron nonlinearities, Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas (2018), https://doi.org/10.1007/s13398-018-0589-6.

Project leader,
conf. dr. Alina Ilinca LAZU